

Quantum mechanics as an asymptotic projection of statistical mechanics of classical fields: derivation of Schrödinger's, Heisenberg's and von Neumann's equations

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Abstract

We show that QM can be represented as a natural projection of a classical statistical model on the phase space $\Omega = H \times H$, where H is the real Hilbert space. Statistical states are given by Gaussian measures on Ω having zero mean value and dispersion of very small magnitude α (which is considered as a small parameter of the model). Such statistical states can be interpreted as fluctuations of the background field, cf. with SED and Nelson's mechanics. Physical variables (e.g., energy) are given by maps $f : \Omega \rightarrow \mathbf{R}$ (functions of classical fields). The conventional quantum representation of our prequantum classical statistical model is constructed on the basis of the Taylor expansion (up to the terms of the second order at the vacuum field point $\psi_{\text{vacuum}} \equiv 0$) of variables $f : \Omega \rightarrow \mathbf{R}$ with respect to the small parameter $\sqrt{\alpha}$. The complex structure of QM is induced by the symplectic structure on the infinite-dimensional phase space Ω . A Gaussian measure (statistical state) is represented in QM by its covariation operator. Equations of Schrödinger, Heisenberg and von Neumann are images of Hamiltonian dynamics on Ω . The main experimental prediction of our prequantum model is that experimental statistical averages can deviate from ones given by QM.

1 Introduction

In the first part of this paper [1] we demonstrated that, in spite of all “NO-GO” theorems, it is possible to construct a general prequantum classical statistical model, cf. with SED [2], [3], Nelson’s stochastic mechanics [4] and Hooft’s deterministic prequantum models [5], [6]. The phase space of this model is the infinite dimensional Hilbert space. Thus classical “systems” are in fact classical fields. We call this approach *Prequantum Classical Statistical Field Theory* (PCSFT). There was constructed a natural map T establishing the correspondence between classical and quantum statistical models. This map T produces the following relation between classical and quantum averages:

$$\langle f \rangle_\rho = \alpha \langle T(f) \rangle_{T(\rho)} + o(\alpha), \quad \alpha \rightarrow 0, \quad (1)$$

where ρ and f are, respectively, a classical statistical state and a classical variable. Here α – the dispersion of the Gaussian measure ρ (having zero mean value) – is considered as a small parameter of the model:

$$\sigma^2(\rho) = \int \|\psi\|^2 d\rho(\psi) = \alpha \rightarrow 0.$$

Quantum states (pure as well as mixed) are images of Gaussian fluctuations of the magnitude α on the infinite dimensional space Ω .

In [1] we considered the quantum model based on the real Hilbert space H . This model is essentially simpler than the complex QM. It is well known that justification of introduction of the complex structure in QM is a very complicated problem. We show that the complex structure is the image of the symplectic structure on the infinite dimensional phase space.

We found the classical Hamiltonian dynamics on the phase space which induces the quantum state dynamics (Schrödinger’s equation). The crucial point is that the classical Hamilton function $\mathcal{H}(\psi)$ should be *J-invariant*:

$$\mathcal{H}(J\psi) = \mathcal{H}(\psi), \quad (2)$$

where $\psi \in \Omega = Q \times P$, $Q = P = H$, and $J : Q \times P \rightarrow Q \times P$ is the symplectic operator. The main reason to consider classical dynamics with *J*-invariant Hamilton functions is that such dynamics preserve the magnitude of classical random fluctuations: the dispersion of a Gaussian measure. In our approach the conventional (linear) quantum dynamics is the image of the classical dynamics for a special class of quadratic Hamilton functions, namely, satisfying

the condition (2). Thus any quantum dynamics is in fact dynamics of a classical (but infinite-dimensional harmonic oscillator). Since all models under consideration are statistical, dynamics of a quantum state (including a pure state) is dynamics of a *Gaussian ensemble of infinite-dimensional harmonic oscillators*. For nonquadratic Hamilton function classical dynamics on Ω can be represented as a nonlinear Schrödinger equation. Thus by representing QM as the image of PCSFT we see that the nonlinear Schrödinger equation is not less natural than the conventional linear equation.

Our approach is based on scaling of the classical prequantum model based on a small parameters $\alpha > 0$. The parameter α describes the magnitude (dispersion) of quantum fluctuations. In our approach quantum averages are obtained as approximations of classical averages (when $\alpha \rightarrow 0$) for amplified classical variables. If for a classical variable $f(\psi)$ we define its *amplification* by

$$f_\alpha(\psi) = \frac{1}{\alpha} f(\psi)$$

then (1) implies that

$$\langle f \rangle_{\text{quantum}} = \lim_{\alpha \rightarrow 0} \langle f_\alpha \rangle_{\text{classical}} \quad (3)$$

In the first version of our approach [1] we identified the parameter α with the Planck constant \hbar (all parameters were considered as dimensionless). This was motivated by SED and Nelson's stochastic QM in that quantum fluctuations have the Planck magnitude. However, our own model does not say anything about relation of the Planck constant and the magnitude of quantum fluctuations. We could not exclude the possibility that the α -scale is essentially finer than the SED-scale based on the Planck constant \hbar .

Thus in our approach QM is a theory about *amplification of quantum fluctuations*, fluctuations of the prequantum classical field ("background field").¹ So we are in the same camp with SED-people with the only possible difference: the energy scale.

We pay attention that any point wise classical dynamics (in particular, Hamiltonian) can be lifted to spaces of variables (functions) and statistical states (probability measures). In the case of a J -invariant Hamilton function

¹It is a good place to cite a remark of Greg Jaeger at the round table of the conference QTRF-3 (Växjö-2005): "Quantum fluctuations are very important. We actually amplify them in our laboratories using Parametric Down Conversion," see [7].

by mapping these lifting to QM we obtain, respectively, Heisenberg's dynamics for quantum observables and von Neumann's dynamics for statistical operators.

We emphasize that one should distinguish (as always in classical statistical physics) dynamics of states of individual physical systems (point wise dynamics) and dynamics of statistical states (dynamics of probability distributions). In conventional QM these two dynamics are typically identified. Our approach supports the original views of E. Schrödinger [8], [9]. Schrödinger's equation is a special type of the Hamiltonian equation on the infinite-dimensional phase-space (the space of classical fields). By our interpretation this equation describes the evolution of classical states (fields). It is impossible to provide any statistical interpretation to such individual states. In particular, the wave function considered as a field satisfying Schrödinger's equation has no statistical interpretation. Only statistical states (probability measures in the classical model) and corresponding density operators (which are in fact scalings of covariation operators of measures representing statistical states) have a statistical interpretation. The root of misunderstanding was assigning (by M. Born) the statistical interpretation to the wave function and at the same time considering it as the complete description of an individual quantum system (the Copenhagen interpretation). The tricky thing is that in fact Born's interpretation should be assigned not to an individual state Ψ , but to a statistical state given by the Gaussian distribution with the covariation operator:

$$B_{\Psi} = \alpha \Psi \otimes \Psi. \quad (4)$$

Thus pure quantum states are simply statistical mixtures of special Gaussian fluctuations (concentrated on two dimensional (real) subspaces of the infinite dimensional Hilbert space), see section 9 for details. One could reproduce dynamics of such a statistical state by considering the Schrödinger equation with random initial conditions:

$$ih \frac{d\xi}{dt}(t; \psi) = \mathbf{H}\xi(t; \psi), \xi(t_0; \psi) = \xi_0(\psi), \quad (5)$$

where \mathbf{H} is Hamiltonian and $\xi_0(\psi)$ is the initial Gaussian random vector taking values in the Hilbert space. We emphasize that $\|\xi(t; \psi)\| \in [0, +\infty)$. There is no place for the standard normalization condition: $\|\xi(t; \psi)\| = 1$. Quantum randomness is not irreducible, cf. with von Neumann [10]. This is classical randomness of initial conditions, cf. with Bohmian mechanics [11], [12].

We remark that the idea that QM can be represented as a probabilistic projection of a classical probabilistic model was elaborated in the series of author's papers, see, e.g., [13]-[17]. In these papers there was introduced *prespace* in that it is possible to provide a finer description of complexes of physical conditions (physical contexts) than in QM. Quantum states were obtained as images of contexts. In the present paper the role of prespace is played by the phase space Ω , contexts are represented by special Gaussian ensembles of classical fields.

Finally, we pay attention that our work might be considered as a realization of "Einstein's dream": creation of purely field model, cf. [18], [19].

2 Hamiltonian mechanics

2.1 Quadratic Hamilton function

We consider the conventional classical phase space:

$$\Omega = Q \times P, \quad Q = P = \mathbf{R}^n$$

Here states are represented by points $\psi = (q, p) \in \Omega$; evolution of a state is described by the Hamiltonian equations

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}, \quad (6)$$

where $\mathcal{H}(q, p)$ is the Hamilton function (a real valued function on the phase space Ω).

We consider the scalar product on $\mathbf{R}^n : (x, y) = \sum_{j=1}^n x_j y_j$ and define the scalar product on $\Omega : (\psi_1, \psi_2) = (q_1, q_2) + (p_1, p_2)$. In our research we shall be interested in quadratic Hamilton functions:

$$\mathcal{H}(q, p) = \frac{1}{2}(\mathbf{H}\psi, \psi), \quad (7)$$

where $\mathbf{H} : \Omega \rightarrow \Omega$ is a symmetric operator. We remark that any (\mathbf{R} -linear) operator $A : \mathbf{R}^{2n} \rightarrow \mathbf{R}^{2n}$ can be represented in the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} : Q \rightarrow Q, A_{12} : P \rightarrow Q, A_{21} : Q \rightarrow P, A_{22} : P \rightarrow P$. A linear operator $A : \mathbf{R}^{2n} \rightarrow \mathbf{R}^{2n}$ is symmetric if

$$A_{11}^* = A_{11}, A_{22}^* = A_{22}, A_{12}^* = A_{21}, A_{21}^* = A_{12}.$$

Thus the Hamilton function (7) can be written as:

$$\mathcal{H}(q, p) = \frac{1}{2}[(\mathbf{H}_{11}q, q) + 2(\mathbf{H}_{12}p, q) + (\mathbf{H}_{22}p, p)], \quad (8)$$

The Hamiltonian equation is linear and it has the form:

$$\dot{q} = \mathbf{H}_{21}q + \mathbf{H}_{22}p, \quad \dot{p} = -(\mathbf{H}_{11}q + \mathbf{H}_{12}p) \quad (9)$$

As always, we define the canonical symplectic structure on the phase space $\Omega = Q \times P$ starting with the *symplectic operator*

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(here the blocks " ± 1 " denote $n \times n$ matrices with ± 1 on the diagonal). By using the symplectic operator J we can write these Hamiltonian equations in the operator form:

$$\dot{\psi} = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J\mathbf{H}\psi \quad (10)$$

Thus

$$\psi(t) = U_t\psi, \quad \text{where } U_t = e^{J\mathbf{H}t}. \quad (11)$$

The map $U_t\psi$ is a linear Hamiltonian flow on the phase space Ω .

2.2 J -invariant quadratic forms and J -commuting operators

In our investigations we shall be concentrated on consideration of J -invariant quadratic forms. It is easy to see that symplectic invariance of the quadratic form $f_A(\psi) = (A\psi, \psi)$, where $A : \Omega \rightarrow \Omega$ is the linear symmetric operator, is equivalent to commuting of A with the symplectic operator J . Let us consider the class $\mathcal{L}_{\text{symp}} \equiv \mathcal{L}_{\text{symp}}(\Omega)$ of linear operators $A : \Omega \rightarrow \Omega$ which commute with the symplectic operator:

$$AJ = JA \quad (12)$$

This is a subalgebra of the algebra of all linear operators $\mathcal{L}(\Omega)$. We call such operators J -commuting.

Proposition 2.1. $A \in \mathcal{L}_{\text{symp}}(\Omega)$ iff $A_{11} = A_{22} = D, A_{12} = -A_{21} = S$:

$$A = \begin{pmatrix} D & S \\ -S & D \end{pmatrix}$$

We remark that an operator $A \in \mathcal{L}_{\text{symp}}(\Omega)$ is symmetric iff $D^* = D$ and $S^* = -S$. Hence any symmetric J -commuting operator in the phase space is determined by a pair of operators (D, S) , where D is symmetric and S is anti-symmetric. Such an operator induces the quadratic form

$$f_A(\psi) = (A\psi, \psi) = (Dq, q) + 2(Sp, q) + (Dp, p). \quad (13)$$

2.3 Dynamics for J -invariant quadratic Hamilton functions

Let us consider an operator $\mathbf{H} \in \mathcal{L}_{\text{symp}}(\Omega)$: $\mathbf{H} = \begin{pmatrix} R & T \\ -T & R \end{pmatrix}$. This operator defines the quadratic Hamiltonian function $\mathcal{H}(q, p) = \frac{1}{2}(\mathbf{H}\psi, \psi)$ which can be written as

$$\mathcal{H}(q, p) = \frac{1}{2}[(Rp, p) + 2(Tp, q) + (Rq, q)] \quad (14)$$

where $R^* = R, T^* = -T$. Corresponding Hamiltonian equations have the form

$$\dot{q} = Rp - Tq, \quad \dot{p} = -(Rq + Tp) \quad (15)$$

Proposition 2.2. *For a J -invariant Hamilton function, the Hamiltonian flow U_t , see (11), consists of J -commuting operators: $U_t J = J U_t$.*

Example 2.1. (One dimensional J -invariant harmonic oscillator) Let $\mathcal{H}(q, p) = \frac{1}{2}[\frac{p^2}{m} + mk^2 q^2]$ (we use the symbol k to denote frequency, since ψ is already used for the point of the phase space). To get a Hamiltonian of the form (14), we consider the case $\frac{1}{m} = mk^2$. Thus $m = \frac{1}{k}$ and $\mathcal{H}(q, p) = \frac{k}{2}[p^2 + q^2]$; Hamiltonian equations are given by $\dot{q} = kp, \quad \dot{p} = -kq$. Here the symmetric J -commuting matrix $\mathbf{H} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

Let us define the *symplectic form* on the phase space:

$$w(\psi_1, \psi_2) = (\psi_1, J\psi_2). \quad (16)$$

Thus

$$w(\psi_1, \psi_2) = (p_2, q_1) - (p_1, q_2)$$

for $\psi_j = \{q_j, p_j\}, j = 1, 2$. This is a *skew-symmetric bilinear form*.

Proposition 2.3. *Let A be a symmetric operator. Then $A \in \mathcal{L}_{\text{symp}}(\Omega)$ iff it is symmetric with respect to the symplectic form:*

$$w(A\psi_1, \psi_2) = w(\psi_1, A\psi_2) \quad (17)$$

Remark 2.1. (Kähler structure) We started our considerations not directly with an arbitrary symplectic form on Ω , but with the canonical symplectic form (16) corresponding to the inner product (Riemannian metric) on Ω . Thus we can canonically introduce the *hermitian metric* on the complex realization Ω_c of Ω . Thus, in fact, from the very beginning we worked not on an arbitrary symplectic manifold, but on a *Kähler manifold*. The J -invariance appears very naturally as the consistency condition for the Riemannian metric and the symplectic structure.

2.4 Complex representation of dynamics for J -invariant quadratic Hamilton functions

Let us introduce on phase space Ω the complex structure: $\Omega_c = Q \oplus iP$. We have $i\psi = -p + iq = -J\psi$. A \mathbf{R} -linear operator $A : \Omega_c \rightarrow \Omega_c$ is \mathbf{C} -linear iff $A(i\psi) = iA\psi$ that is equivalent to $A \in \mathcal{L}_{\text{symp}}(\Omega)$.

Proposition 2.4. *The class of \mathbf{C} -linear operators $\mathcal{L}(\Omega_c)$ coincides with the class of J -commuting operators $\mathcal{L}_{\text{symp}}(\Omega)$.*

We introduce on Ω_c a complex scalar product (hermitian metric, see Remark 2.1) based on the \mathbf{C} -extension of the real scalar product:

$$\begin{aligned} \langle \psi_1, \psi_2 \rangle &= \langle q_1 + ip_1, q_2 + ip_2 \rangle \\ &= (q_1, q_2) + (p_1, p_2) + i((p_1, q_2) - (p_2, q_1)). \end{aligned}$$

Thus

$$\langle \psi_1, \psi_2 \rangle = (\psi_1, \psi_2) - iw(\psi_1, \psi_2),$$

where w is the symplectic form. This is the canonical hermitian metric on the Kähler manifold Ω .

A \mathbf{C} -linear operator A is symmetric with respect to the complex scalar product $\langle \dots \rangle$ iff it is symmetric with respect to both real bilinear forms:

(\cdot, \cdot) and $w(\cdot, \cdot)$. Since for $A \in \mathcal{L}_{\text{symp}}(\Omega)$ the former implies the latter, we get that a \mathbf{C} -linear operator in Ω_c is symmetric iff it is symmetric in the real space Ω .

Proposition 2.5. *The class of \mathbf{C} -linear symmetric operators $\mathcal{L}_s(\Omega_c)$ coincides with the class of J -commuting symmetric operators $\mathcal{L}_{\text{symp},s}(\Omega)$.*

We also remark that for a J -commuting operator A its *real and complex adjoint operators*:

$$A^\star \text{ and } A^*$$

coincide. We showed that \mathbf{C} -linear symmetric operators appear naturally as complex representations of J -commuting symmetric operators.

Proposition 2.6. *For a quadratic J -invariant Hamilton function $\mathcal{H}(\psi)$, its complexification does not change dynamics.*

Proof. To prove this, we remark that $w(\mathbf{H}\psi, \psi) = 0$ and hence

$$\mathcal{H}(\psi) = \frac{1}{2} \langle \mathbf{H}\psi, \psi \rangle = \frac{1}{2} [(\mathbf{H}\psi, \psi) - iw(\mathbf{H}\psi, \psi)] = \frac{1}{2} (\mathbf{H}\psi, \psi), \psi \in \Omega.$$

I consider the introduction of a complex structure on the phase-space merely as using a new language: instead of symplectic invariance, we speak about \mathbf{C} -linearity. By Proposition 2.6 the Hamilton function (14) can be written $\mathcal{H}(\psi) = \frac{1}{2} \langle \mathbf{H}\psi, \psi \rangle$, $\mathbf{H} \in \mathcal{L}_s(\mathbf{C}^n)$, and the Hamiltonian equation (10) can be written in the complex form as:

$$i \frac{d\psi}{dt} = \mathbf{H}\psi \tag{18}$$

Any solution has the following complex representation:

$$\psi(t) = U_t \psi, \quad U_t = e^{-i\mathbf{H}t/\hbar}. \tag{19}$$

This is the complex representation of flows corresponding to quadratic J -invariant Hamilton functions.

3 Schrödinger dynamics as a dynamics with J -invariant Hamilton function on the infinite dimensional phase space

Let Ω_c be a complex Hilbert space (infinite dimensional and separable) and let $\langle \cdot, \cdot \rangle$ be the complex scalar product on Ω_c . The symbol $\mathcal{L}_s \equiv \mathcal{L}_s(\Omega_c)$

denotes the space of continuous \mathbf{C} -linear self-adjoint operators. We use the Planck system of units: $\hbar = 1$. The Schrödinger dynamics in Ω is given by the linear equation:

$$i\frac{d\psi}{dt} = \mathbf{H}\psi \quad (20)$$

and hence

$$\psi(t) = U_t\psi, \quad U_t = e^{-i\mathbf{H}t}. \quad (21)$$

We see that these are simply infinite-dimensional versions of equations (18) and (19) obtained from the Hamiltonian equations for a quadratic J -invariant Hamilton function in the process of complexification of classical mechanics. Therefore we can reverse our previous considerations (with the only remark that now the phase space is infinite dimensional) and represent the Schrödinger dynamics (20) in the complex Hilbert space as the Hamiltonian dynamics in the infinite-dimensional phase space.² We emphasize that this Hamiltonian dynamics (10) is a dynamics in the phase space Ω and not in the unit sphere of this Hilbert space! The Hamiltonian flow $\psi(t, \psi) = U_t\psi$ is a flow on the whole phase space Ω .

We consider in Ω the \mathbf{R} -linear operator J corresponding to multiplication by $-i$; we represent the complex Hilbert space in the form:

$$\Omega_c = Q \oplus iP,$$

where Q and P are copies of the real Hilbert space. Here $\psi = q + ip$. We emphasize that q and p are not ordinary position and momentum for particles. These are their field analogues (if we choose $Q = P = L_2(\mathbf{R}^3)$): these are functions of $x \in \mathbf{R}^3$. We consider now the real phase space:

$$\Omega = Q \times P.$$

As in the finite dimensional case, we have:

Proposition 3.1. *The class of continuous \mathbf{C} -linear self-adjoint operators $\mathcal{L}_s(\Omega_c)$ coincides with the class of continuous J -commuting self-adjoint operators $\mathcal{L}_{\text{symp},s}(\Omega)$.*

²Infinite dimension induces merely mathematical difficulties. The physical interpretation of formalism is the same as in the finite-dimensional case.

Let us consider a quantum Hamiltonian $\mathbf{H} \in \mathcal{L}_s(\Omega_c)$.³ It determines the classical Hamiltonian function:

$$\mathcal{H}(\psi) = \frac{1}{2} \langle \mathbf{H}\psi, \psi \rangle = \frac{1}{2} [(Rp, p) + 2(Tp, q) + (Rq, q)]$$

The corresponding Hamiltonian equation on the classical phase space $\Omega = Q \times P$, where Q and P are copies of the real Hilbert space, is given by

$$\frac{dq}{dt} = Rp - Tq, \quad \frac{dp}{dt} = -(Rq + Tp) \quad (22)$$

If we apply the complexification procedure to this system of Hamiltonian equations we, of course, obtain the Schrödinger equation (20).

Example 3.1. Let us consider an important class of Hamilton functions

$$\mathcal{H}(q, p) = \frac{1}{2} [(Rp, p) + (Rq, q)], \quad (23)$$

where R is a symmetric operator. The corresponding Hamiltonian equations have the form:

$$\dot{q} = Rp, \quad \dot{p} = -Rq. \quad (24)$$

We now choose $H = L_2(\mathbf{R}^3)$, so $q(x)$ and $p(x)$ are components of the vector-field $\psi(x) = (q(x), p(x))$. We can call fields $q(x)$ and $p(x)$ *mutually inducing*. The presence of the field $p(x)$ induces dynamics of the field $q(x)$ and vice versa, cf. with electric and magnetic components, $q(x) = E(x)$ and $p(x) = B(x)$, of the classical electromagnetic field, cf. Einstein and Infeld [19], p. 148: “Every change of an electric field produces a magnetic field; every change of this magnetic field produces an electric field; every change of ..., and so on.” We can write the form (23) as

$$\mathcal{H}(q, p) = \frac{1}{2} \int_{\mathbf{R}^6} R(x, y) [q(x)q(y) + p(x)p(y)] dx dy \quad (25)$$

or

$$\mathcal{H}(\psi) = \frac{1}{2} \int_{\mathbf{R}^6} R(x, y) \psi(x) \bar{\psi}(y) dx dy, \quad (26)$$

where $R(x, y) = R(y, x)$ is in general a distribution on \mathbf{R}^6 .

³We may consider operator $\mathbf{H} \geq 0$, but for the present consideration this is not important.

We call such a kernel $R(x, y)$ a *self-interaction potential* for the field $\psi(x) = (q(x), p(x))$. We pay attention that $R(x, y)$ induces a self-interaction of each component of the $\psi(x)$, but there is no cross-interaction between components $q(x)$ and $p(x)$ of the vector-field $\psi(x)$.

One may justify consideration of J -invariant physical variables on the Hilbert phase space by referring to quantum mechanics: “the correct classical Hamiltonian dynamics is based on J -invariant Hamilton functions, because they induce the correct quantum dynamics.” So the classical prequantum dynamics was reconstructed on the basis of the quantum dynamics. I have nothing against such an approach. But it would be interesting to find internal classical motivation for considering J -invariant Hamilton functions. We shall do this in section 5.

4 Lifting of point wise dynamics to spaces of variables and measures

4.1 General dynamical framework

Let (X, F) be an arbitrary measurable space. Here X is a set and F is a σ -field of its subsets. Denote the space of random variables (measurable maps $f : X \rightarrow \mathbf{R}$) by the symbol $RV(X)$ and the space of probability measures on (X, F) by the symbol $PM(X)$. Consider a measurable map $g : X \rightarrow X$. It induces the maps

$$g^* : RV(X) \rightarrow RV(X), \quad g^*f(x) = f(g(x))$$

$$g^* : MP(X) \rightarrow MP(X), \quad \int_X f(x) dg^*\mu(x) = \int_X g^*f(x) d\mu(x).$$

We now consider a dynamical system in X :

$$x_t = g_t(x), \tag{27}$$

where $g_t : X \rightarrow X$ is an one-parametric family of measurable maps (the parameter t is real and plays the role of time). By using lifting α and β we can lift this point wise dynamics in X to dynamics in $RV(X)$ and $MP(X)$, respectively:

$$f_t = g_t^*f \tag{28}$$

$$\mu_t = g_t^* \mu. \quad (29)$$

We shall see in sections 6, 7 that for $X = \Omega$ (infinite dimensional phase space) quantum images of dynamical systems (27), (28), (29) are respectively dynamics of Schrödinger (for state – wave function), Heisenberg (for operators-observables) and von Neumann (for density operator). To obtain quantum mechanics, we should choose adequate spaces of physical variables and measures.

4.2 Lifting of the Hamiltonian dynamics

It is well known that the lifting of Hamiltonian dynamics to the space of smooth variables is given by the *Liouville equation*, see e.g. [20]. In particular, the functional lifting of any Hamiltonian dynamics on the Hilbert phase space Ω can be represented as the infinite-dimensional Liouville equation, [21]. We remark that this is a general fact which has no relation to our special classical framework based on J -invariant Hamilton functions. For smooth functions on the Ω we introduce the Poisson brackets, see, e.g., [22]:

$$\{f_1(\psi), f_2(\psi)\} = \left(\frac{\partial f_1}{\partial q}(\psi), \frac{\partial f_2}{\partial p}(\psi) \right) - \left(\frac{\partial f_2}{\partial q}(\psi), \frac{\partial f_1}{\partial p}(\psi) \right).$$

We recall that for $f : H \rightarrow \mathbf{R}$ its first derivative can be represented as a vector belonging H ; so for $f : H \times H \rightarrow \mathbf{R}$ its gradient $\nabla f(\psi)$ belongs $H \times H$. We pay attention that $\{f_1, f_2\} = (\nabla f_1, J \nabla f_2) = w(\nabla f_1, \nabla f_2)$. Let $\mathcal{H}(\psi)$ be a smooth Hamilton function inducing the flow $U_t(\psi)$. For a smooth function f_0 we set $f(t, \psi) = f_0(U_t(\psi))$. It is easy to see that this function is the solution of the Cauchy problem for the Liouville equation:

$$\frac{\partial f}{\partial t}(t, \psi) = \{f(t, \psi), \mathcal{H}(\psi)\}, \quad f(0, \psi) = f_0(\psi) \quad (30)$$

The functional flow $\Psi(t, f_0) = \alpha_{U_t} f_0$ can be represented as

$$\Psi(t, f_0) = e^{-tL} f_0, \quad (31)$$

where

$$L = \left(\frac{\partial \mathcal{H}}{\partial q}(\psi), \frac{\partial}{\partial p} \right) - \left(\frac{\partial \mathcal{H}}{\partial p}(\psi), \frac{\partial}{\partial q} \right)$$

5 Dispersion preserving dynamics of statistical states

Everywhere in this section we consider only quadratic Hamilton functions on the infinite-dimensional phase space Ω . We start our consideration with an arbitrary quadratic Hamiltonian function $\mathcal{H}(\psi) = \frac{1}{2}(\mathbf{H}\psi, \psi)$ (the operator \mathbf{H} need not be J -commuting). Let us consider the Hamiltonian flow $U_t : \Omega \rightarrow \Omega$ induced by the Hamiltonian system (10). This map is given by (11). It is important to pay attention that the map U_t is invertible; in particular,

$$U_t(\Omega) = \Omega. \quad (32)$$

We are interested in a Hamiltonian flow U_t such that the corresponding dynamics in the space of probabilities (29) preserves the magnitude of statistical fluctuations:

$$\sigma^2(U_t^* \rho) = \sigma^2(\rho) : \int_{\Omega} \|\psi\|^2 dU_t^* \rho(\psi) = \int_{\Omega} \|\psi\|^2 d\rho(\psi) \quad (33)$$

or

$$\int_{\Omega} \|U_t \psi\|^2 d\rho(\psi) = \int_{\Omega} \|\psi\|^2 d\rho(\psi). \quad (34)$$

We start the study of this problem with a sufficient condition for preserving the magnitude of statistical fluctuations: the Hamiltonian flow $U_t \psi$ consists of isometric maps:

$$\|U_t \psi\|^2 = \|\psi\|^2, \psi \in \Omega. \quad (35)$$

Proposition 5.1. *A Hamiltonian flow $U_t \psi$ is isometric iff the function $\mathcal{H}(\psi)$ is J -invariant.*

Proof. a). Let \mathcal{H} be J -commuting. Then we have:

$$\frac{d}{dt} \|U_t \psi\|^2 = 2(\dot{U}_t \psi, U_t \psi) = 2(J\mathbf{H}U_t \psi, U_t \psi) = 0$$

Here we used the simple fact that the operator $J\mathbf{H}$ is skew symmetric: $(J\mathbf{H})^* = -\mathbf{H}J = -J\mathbf{H}$. Thus (35) holds.

b). Let (35) hold. Then $\frac{d}{dt} \|U_t \psi\|^2 = 0$. By using previous computations and (32) we get that:

$$(J\mathbf{H}\psi, \psi) = 0, \psi \in \Omega. \quad (36)$$

Hence the operator $J\mathbf{H}$ is skew symmetric. This implies that \mathbf{H} commutes with J .

For our further considerations, see section 12, it is useful to rewrite (36) in the form:

$$(J\mathcal{H}'(\psi), \psi) = 0, \psi \in \Omega. \quad (37)$$

Corollary 5.1. *The flow corresponding to a J -invariant Hamilton function preserves the fluctuations of the fixed α -magnitude. For any measure ρ (having the zero mean value and finite dispersion), if $\sigma^2(\rho) = \alpha$, then $\sigma^2(U_t^* \rho) = \alpha$ for any $t \geq 0$.*

This is our explanation of the exceptional role of J -invariant physical variables on the infinite-dimensional classical phase space.

If a Hamilton function is not J -invariant then the corresponding Hamiltonian flow can induce increasing of the magnitude of fluctuations. But we recall that quantum model is a representation based on neglecting by fluctuations of the magnitude $o(\alpha)$, $\alpha \rightarrow 0$. Therefore a Hamiltonian flow which is not J -invariant can induce the transformation of “quantum statistical states”, i.e., distributions on the phase space having dispersion of the magnitude α , into “nonquantum statistical states”, i.e. distributions on the phase space having dispersions essentially larger than α .

6 Dynamics in the space of physical variables

6.1 Arbitrary quadratic variables

Let us consider the Hamiltonian flow $U_t : \Omega \rightarrow \Omega$ induced by an arbitrary quadratic Hamilton function. Let $A : \Omega \rightarrow \Omega$ be a continuous self-adjoint operator and $f_A = (A\psi, \psi)$. We have $U_t^* f_A(\psi) = f_A(U_t \psi) = f_{U_t^* A U_t}(\psi)$. This dynamics can be represented as the dynamics in the space of continuous linear symmetric operators

$$A_t = U_t^* A U_t \quad (38)$$

We remark that $U_t = e^{J\mathbf{H}t}$, so $U_t^* = e^{-\mathbf{H}Jt}$. Thus

$$A_t = e^{-\mathbf{H}Jt} A e^{J\mathbf{H}t}. \quad (39)$$

Thus $\frac{dA_t}{dt} = (A_t J\mathbf{H} - \mathbf{H}J A_t)$, or

$$\frac{dA_t}{dt} = [A_t, \mathbf{H}J] + A_t[J, \mathbf{H}] \quad (40)$$

We remark that dynamics (38) can be also obtained from the Liouville equation, but I presented the direct derivation.

6.2 J -invariant variables

We consider the space of physical variables

$$V_{\text{quad,symp}}(\Omega) = \{f : \Omega \rightarrow \mathbf{R} : f \equiv f_A(\psi) = \frac{1}{2}(A\psi, \psi), A \in \mathcal{L}_{\text{symp,s}}(\Omega)\}$$

(consisting of J -invariant quadratic forms). Let us consider the lifting of the flow corresponding to a J -invariant quadratic Hamilton function to the space $V_{\text{quad,symp}}(\Omega)$. In this case both operators, \mathbf{H} and A are J -commuting. Therefore the flow (39) can be written as

$$A_t = U_t^* A U_t = e^{-J\mathbf{H}t} A e^{J\mathbf{H}t} \quad (41)$$

The evolution equation (40) is simplified:

$$\frac{dA_t}{dt} = -J[\mathbf{H}, A_t] \quad (42)$$

6.3 Complexification

As in section 6.2, we suppose that $[\mathbf{H}, J] = 0$ and $[A, J] = 0$. By considering on the phase space the complex structure and representing the symplectic operator J by $-i$ we write (39) in the form of the Heisenberg dynamics:

$$A_t = U_t^* A U_t = e^{it\mathbf{H}} A e^{-it\mathbf{H}} \quad (43)$$

(here U_t^* is the complex adjoint operator to U_t) and the evolution equation (40) in the form of the Heisenberg equation:

$$\frac{dA_t}{dt} = i[\mathbf{H}, A_t] \quad (44)$$

Thus this equation is just the image of the lifting of the classical quadratic Hamiltonian dynamics in the case of J -invariant quadratic variables.

7 Dynamics in the space of statistical states

7.1 Arbitrary Gaussian measures

Let us consider the flow $U_t : \Omega \rightarrow \Omega$ induced by an arbitrary quadratic Hamilton function $\mathcal{H}(\psi)$. Let ρ be an arbitrary Gaussian measure with zero mean value. Since a linear continuous transformation of a Gaussian measure is again a Gaussian measure, we have that $U_t^*(\rho)$ is Gaussian. We find dynamics of the covariation operator of $U_t^*(\rho)$. We have:

$$\begin{aligned} (\text{cov}(U_t^*\rho)y_1, y_2) &= \int_{\Omega} (y_1, \psi)(y_2, \psi) dU_t^*\rho(\psi) \\ &= \int_{\Omega} (y_1, U_t\psi)(y_2, U_t\psi) d\rho(\psi) = (\text{cov}(\rho)U_t^*y_1, U_t^*y_2). \end{aligned}$$

Thus, for the covariation operator $B_t = \text{cov}(U_t^*\rho)$, we have:

$$B_t = U_t B U_t^* \equiv e^{J\mathbf{H}t} B e^{-\mathbf{H}Jt} \quad (45)$$

Thus $\frac{dB_t}{dt} = (J\mathbf{H}B_t - B_t\mathbf{H}J)$, or

$$\frac{dB_t}{dt} = [J\mathbf{H}, B_t] + B_t[J, \mathbf{H}] \quad (46)$$

7.2 J -invariant measures

We now consider the lifting (to the space of measures) of the flow $U_t : \Omega \rightarrow \Omega$ induced by a J -invariant quadratic Hamilton function $\mathcal{H}(\psi)$. We start with the following mathematical result:

Proposition 7.1. *A Gaussian measure ρ (with zero mean value) is J -invariant iff its covariation operator is J -invariant.*

Proof. a). Let $J^*\rho = \rho$. It is sufficient to prove that BJ is skew symmetric, where $B = \text{cov } \rho$. We have:

$$\begin{aligned} (BJy_1, y_2) &= \int_{\Omega} (Jy_1, \psi)(y_2, \psi) d\rho(\psi) = - \int_{\Omega} (y_1, J\psi)(y_2, J^*J\psi) d\rho(\psi) \\ &= - \int_{\Omega} (y_1, J\psi)(Jy_2, J\psi) d\rho(\psi) = - \int_{\Omega} (y_1, \psi)(Jy_2, \psi) d\beta_{J\rho}(\psi) \end{aligned}$$

$$= - \int_{\Omega} (Jy_2, \psi)(y_1, \psi) d\rho(\psi) = -(BJy_2, y_1) = -(y_1, BJy_2).$$

b). Let $B = \text{cov}(\rho) \in \mathcal{L}_{\text{symp},s}(\Omega)$. We find the Fourier transform of the Gaussian measure $J^*\rho$:

$$\widetilde{J^*\rho}(y) = \int_{\Omega} e^{i(y, J\psi)} d\rho(\psi) == \tilde{\rho}(J^*y) = e^{-\frac{1}{2}(BJ^*y, J^*y)} == \tilde{\rho}(y).$$

From the proof we also obtain:

Corollary 7.1. *Let ρ be an arbitrary J -invariant measure. Then its covariation operator is J -invariant.*

Since the flow for a J -invariant quadratic Hamilton function consists of J -commuting linear operators, $JU_t = U_tJ$, by using the representation (45) and Proposition 7.1 we prove that the space of J -invariant Gaussian measures (with zero mean value) is invariant for the map U_t^* . Here we have:

$$B_t = U_t B U_t^* \equiv e^{J\mathbf{H}t} B e^{-J\mathbf{H}t} \quad (47)$$

or

$$\frac{dB_t}{dt} = -J[B_t, \mathbf{H}] \quad (48)$$

7.3 Complexification

We suppose that $[\mathbf{H}, J] = 0$ and $[B, J] = 0$. By considering on the phase space the complex structure and representing the symplectic operator J by $-i$ we write (47) in the form:

$$B_t = U_t B U_t^* = e^{-i\mathbf{H}t} B e^{i\mathbf{H}t} \quad (49)$$

or

$$\frac{dB_t}{dt} = i[B_t, \mathbf{H}] \quad (50)$$

This is nothing else than the von Neumann equation for the statistical operator. The only difference is that the covariance operator B is not normalized. The normalization will come from the correspondence map T projecting a prequantum classical statistical model onto QM, see section 8.

7.4 Dynamics in the space of statistical states

First we consider the space of all Gaussian measures having zero mean value and dispersion α . We recall that here α is a small parameter characterizing fluctuations of energy of the background field:

$$\alpha = \int_{L_2(\mathbf{R}^3) \times L_2(\mathbf{R}^3)} \int_{\mathbf{R}^3} (|q(x)|^2 + |p(x)|^2) dx d\rho(q, p).$$

We do not provide dimension analysis in this paper. But the crucial point is that elements of the phase space $\Omega = L_2(\mathbf{R}^3) \times L_2(\mathbf{R}^3)$, “wave functions”, are considered as classical fields (as the classical electromagnetic field) and not as ‘square roots of probabilities’ (cf. with the conventional Born’s interpretation of the wave function, but also cf. with the original Schrödinger’s interpretation).

Denote this space of such measures by the symbol $S_G^\alpha(\Omega)$. These are Gaussian measures such that

$$(y, m_\rho) = \int_{\Omega} (y, \psi) d\rho(\psi) = 0, y \in \Omega, \text{ and } \sigma^2(\rho) = \int_{\Omega} \|\psi\|^2 d\rho(\psi) = \alpha$$

For the flow U_t corresponding to a J -invariant quadratic Hamilton function, we have (see section 5)

$$U_t^* : S_G^\alpha(\Omega) \rightarrow S_G^\alpha(\Omega)$$

Denote the subspace of $S_G^\alpha(\Omega)$ consisting of J -invariant measures by the symbol $S_{G,\text{symp}}^\alpha(\Omega)$. We also have:

$$U_t^* : S_{G,\text{symp}}^\alpha(\Omega) \rightarrow S_{G,\text{symp}}^\alpha(\Omega).$$

7.5 Complex covariation

Everywhere below we consider only measures with finite dispersions. Let us introduce *complex average* m_ρ^c and *covariance operator* $B^c \equiv \text{cov}^c \rho$ by setting:

$$\langle m_\rho^c, y \rangle = \int_{\Omega} \langle y, \psi \rangle d\rho(\psi). \quad (51)$$

$$\langle B^c y_1, y_2 \rangle = \int_{\Omega} \langle y_1, \psi \rangle \langle \psi, y_2 \rangle d\rho(\psi). \quad (52)$$

Proposition 7.2. *Let ρ be a J -invariant measure. Then*

$$m_\rho^c = 0 \text{ iff } m_\rho = 0. \quad (53)$$

Proof. Since ρ is J -invariant, for any Borel function $f : \Omega \rightarrow \mathbf{R}$, we have:

$$\int_{\Omega} f(\psi_q, \psi_p) d\rho(\psi_q, \psi_p) = \int_{\Omega} f(\psi_p, -\psi_q) d\rho(\psi_q, \psi_p) \quad (54)$$

Let $m_\rho = 0$. Then:

$$\begin{aligned} 0 &= \int_{\Omega} (y, \psi) d\rho(\psi) = \int_{\Omega} [(y_q, \psi_q) + (y_p, \psi_p)] d\rho(\psi) \\ &= \int_{\Omega} [(y_q, \psi_p) - (y_p, \psi_q)] d\rho(\psi) = \int_{\Omega} w(y, \psi) d\rho(\psi), \end{aligned}$$

where w is the symplectic form Ω . Hence the last integral is also equal to zero. On the other hand, for the complex average we have:

$$\langle y, m_\rho^c \rangle = \int_{\Omega} (y, \psi) d\rho(\psi) - i \int_{\Omega} w(y, \psi) d\rho(\psi). \quad (55)$$

Proposition 7.3. *Let ρ be an arbitrary J -invariant measure with the zero mean value. Then*

$$\text{cov}^c \rho = 2 \text{cov } \rho \quad (56)$$

Proof. We have

$$\begin{aligned} \text{cov}^c \rho(y, y) &= \int_{\Omega} |\langle y, \psi \rangle|^2 d\rho(\psi) = \int_{\Omega} |(y, \psi) - iw(y, \psi)|^2 d\rho(\psi) \\ &= \int_{\Omega} [(y, \psi)^2 + (y, J\psi)^2] d\rho(\psi). \end{aligned}$$

By using symplectic invariance of the measure ρ we get:

$$\int_{\Omega} (y, J\psi)^2 d\rho(\psi) = \int_{\Omega} (y, \psi)^2 d\rho(\psi).$$

Thus

$$\text{cov}^c \rho(y, y) = 2 \int_{\Omega} (y, \psi)^2 d\rho(\psi) = 2 \text{cov} \rho(y, y).$$

Theorem 7.1. *For any measure ρ with the zero mean value and any J -commuting operator A , we have:*

$$\int_{\Omega} \langle A\psi, \psi \rangle d\rho(\psi) = \text{Tr cov}^c \rho A; \quad (57)$$

in particular,

$$\sigma^2(\rho) = \text{Tr cov}^c \rho. \quad (58)$$

Proof. Let $\{e_j\}$ be an orthonormal basis in Ω_c (we emphasize that orthogonality and normalization are with respect to the complex and not real scalar product). Then:

$$\text{Tr cov}^c \rho A = \int_{\Omega} \sum_j \langle Ae_j, \psi \rangle \langle \psi, e_j \rangle d\rho(\psi) = \int_{\Omega} \langle A\psi, \psi \rangle d\rho(\psi).$$

We recall that we showed in [1] that and

$$\sigma^2(\rho) = \text{Tr cov} \rho. \quad (59)$$

It seems that there is a contradiction between equalities (59), (58) and (56). In fact, there is no contradiction, because in (59) and (58) we use two different traces: with respect to the real and complex scalar products, respectively. This is an important point; even normalization by trace one for the von Neumann density operator is the normalization with respect to the complex scalar product. By using indexes **R** and **C** to denote real and complex traces, respectively, we can write:

$$\sigma^2(\rho) = \text{Tr}_{\mathbf{R}} \text{cov} \rho = \text{Tr}_{\mathbf{C}} \text{cov}^c \rho.$$

We remark that *the complex average m_{ρ}^c and the covariation operator B^c are **C**-linear even if a measure is not J -invariant*. However, in general real and complex averages do not coincide and real and complex covariance operators are not coupled by (56).

Let us find relation between $B = \text{cov} \rho$ and $B^c = \text{cov}^c \rho$ in the general case. It is easy to see that for

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, B_{11}^* = B_{11}, B_{22}^* = B_{22}, B_{12}^* = B_{21}$$

and

$$B^c = \begin{pmatrix} D & S \\ -S & D \end{pmatrix}$$

we have

Proposition 7.4. *The blocks in real and complex covariation operators are connected by the following equalities:*

$$D = B_{11} + B_{22}, S = B_{12} - B_{21}. \quad (60)$$

Thus in the general case the complex covariation operator B^c does not determine the Gaussian measure ρ_B uniquely.

Let now ρ_B be J -invariant. Then

$$B = \begin{pmatrix} B_{11} & B_{12} \\ -B_{12} & B_{11} \end{pmatrix}.$$

Thus

$$D = 2B_{11}, S = 2B_{12}, \quad (61)$$

so we obtain (56) and, hence, we obtain:

Corollary 7.2. *There is one-to-one correspondence between J -invariant Gaussian measures with the zero mean value and complex covariation operators.⁴*

As was remarked, even for a Gaussian measure ρ which is not J -invariant its complex covariation operator B^c does not define ρ uniquely. Nevertheless, let us represent an arbitrary measure ρ (with zero mean value and finite dispersion) by its complex covariation operator B^c (so we “project” measures to their complex covariation operators).

Let us consider the dynamics of ρ induced by a dynamics in Ω with a quadratic J -invariant Hamilton function \mathcal{H} . We obtain a one-parameter family of measures $\rho_t = U_t^* \rho$. It is easy to see that $B^c(t) = U_t B^c U_t^*$. Since $[B^c, J] = 0$, the $B^c(t)$ satisfies the von Neumann equation (50).

8 Prequantum classical statistical model

We consider the infinite-dimensional phase-space (space of classical fields) $\Omega = Q \times P$, where Q and P are copies of the (separable) Hilbert space. Our

⁴These are \mathbf{C} -linear self-adjoint positively defined operators $B^c : \Omega_c \rightarrow \Omega_c$ belonging to the trace class

aim is to construct a prequantum classical statistical model on this phase-space inducing the conventional (Dirac-von Neumann) quantum statistical model

$$N_{\text{quant}} = (\mathcal{D}(\Omega_c), \mathcal{L}_s(\Omega_c)),$$

where the complex Hilbert space $\Omega_c = Q \oplus iP$. Here $\mathcal{D}(\Omega_c)$ is the space of density operators and $\mathcal{L}_s(\Omega_c)$ is the space of bounded self-adjoint operators in Ω_c (quantum observables).⁵

We choose the space of classical statistical states $S_{G,\text{symp}}^\alpha(\Omega)$ consisting of J -invariant Gaussian measures having zero mean value and dispersion α .

We choose, cf. [1], the space of classical physical variables as the functional space $\mathcal{V}_{\text{symp}}(\Omega)$ consisting of real analytic functions, $f : \Omega \rightarrow \mathbf{R}$, that have the exponential growth:

$$\text{there exist } C_0, C_1 \geq 0 : |f(\psi)| \leq C_0 e^{C_1 \|\psi\|}; \quad (62)$$

preserve the state of vacuum:

$$f(0) = 0 \quad (63)$$

and that are J -invariant: $f(J\psi) = f(\psi)$.

We pay attention that any $f \in \mathcal{V}_{\text{symp}}(\Omega)$ is an *even function*: $f(-\psi) = f(J^2\psi) = f(J\psi) = f(\psi)$. We shall also use a simple consequence of this result: if $f \in \mathcal{V}_{\text{symp}}(\Omega)$, then its derivative is an odd function.

Example 8.1. Let $\mathbf{H} \in \mathcal{L}_{\text{symp},s}(\Omega)$. Then any polynomial $f(\psi) = \sum_{k=1}^N a_k (\mathbf{H}\psi, \psi)^k$, $a_k \in \mathbf{R}$, belongs to the space $\mathcal{V}_{\text{symp}}(\Omega)$.

The following trivial mathematical result plays the fundamental role in establishing classical \rightarrow quantum correspondence.

Proposition 8.1. *Let $f \in \mathcal{V}_{\text{symp}}(\Omega)$. Then*

$$f''(0) \in \mathcal{L}_{\text{symp},s}(\Omega). \quad (64)$$

We remark that for an arbitrary $\psi \in \Omega$ we have

$$Jf''(\psi) = f''(J\psi)J. \quad (65)$$

⁵To simplify considerations, we consider only quantum observables represented by bounded operators. To obtain the general quantum model with observables represented by unbounded operators, we should consider a prequantum classical statistical model based on the Gelfand triple: $\Omega_c^+ \subset \Omega_c \subset \Omega_c^-$.

We consider now the classical statistical model:

$$M_{\text{symp}}^\alpha = (S_{G,\text{symp}}^\alpha(\Omega), \mathcal{V}_{\text{symp}}(\Omega)). \quad (66)$$

Let us find the average of a variable $f \in \mathcal{V}_{\text{symp}}(\Omega)$ with respect to a statistical state $\rho_B \in S_{G,\text{symp}}^\alpha(\Omega)$:

$$\begin{aligned} \langle f \rangle_{\rho_B} &= \int_{\Omega} f(\psi) d\rho_B(\psi) = \int_{\Omega} f(\sqrt{\alpha}\psi) d\rho_D(\psi) \\ &= \sum_{n=2}^{\infty} \frac{(\alpha)^{n/2}}{n!} \int_{\Omega} f^{(n)}(0)(\psi, \dots, \psi) d\rho_D(\psi), \end{aligned} \quad (67)$$

where the covariation operator of the $\sqrt{\alpha}$ -scaling ρ_D of the Gaussian measure ρ_B has the form:

$$D = B/\alpha.$$

Since $\rho_B \in S_G^\alpha(\Omega)$, we have $\text{Tr } D = 1$. The change of variables in (67) can be considered as scaling of the magnitude of statistical (Gaussian) fluctuations. Fluctuations which were considered as very small,

$$\sigma^2(\rho) = \alpha, \quad (68)$$

(where $\alpha \rightarrow 0$ is a small parameter) are considered in the new scale as standard normal fluctuations.⁶ By (67) we have:

$$\langle f \rangle_{\rho} = \frac{\alpha}{2} \int_{\Omega} (f''(0)\psi, \psi) d\rho_D(\psi) + o(\alpha), \quad \alpha \rightarrow 0, \quad (69)$$

or

$$\langle f \rangle_{\rho} = \frac{\alpha}{2} \text{Tr } D f''(0) + o(\alpha), \quad \alpha \rightarrow 0. \quad (70)$$

Finally, we rewrite the formulas (69) and (70) in the complex form:

$$\langle f \rangle_{\rho} = \frac{\alpha}{2} \int_{\Omega} \langle f''(0)\psi, \psi \rangle d\rho_D(\psi) + o(\alpha), \quad \alpha \rightarrow 0, \quad (71)$$

or

$$\langle f \rangle_{\rho} = \alpha \text{Tr } D^c \frac{f''(0)}{2} + o(\alpha), \quad \alpha \rightarrow 0. \quad (72)$$

⁶Thus QM is a kind of the statistical microscope which gives us the possibility to see the effect of fluctuations of the magnitude α in a neighborhood of vacuum field point, ($\psi_{\text{vacuum}} \equiv 0$).

We pay attention that in (70) a trace is the trace with respect to the real scalar product and in (72) - the complex scalar product.

For a classical variable $f(\psi)$, we define its amplification by

$$f_\alpha(\psi) = \frac{1}{\alpha} f(\psi)$$

(when $\alpha \rightarrow 0$ this amplification will be becoming infinitely large). We see that the classical average of the amplification $f_\alpha(\psi)$ of a classical variable $f(\psi)$ (computed in the model (66) by using the measure-theoretic approach) is approximately equal to the quantum average (computed in the model $N_{\text{quant}} = (\mathcal{D}(\Omega_c), \mathcal{L}_s(\Omega_c))$ with the aid of the von Neumann trace-formula):

$$\langle f_\alpha \rangle_\rho = \text{Tr } D^c \frac{f''(0)}{2} + o(1), \quad \alpha \rightarrow 0. \quad (73)$$

The classical \rightarrow quantum correspondence map T is similar to the map presented in [1] in the real case:

$$T : S_{G, \text{symp}}^\alpha(\Omega) \rightarrow \mathcal{D}(\Omega_c), \quad T(\rho) = \frac{\text{cov}^c \rho}{\alpha} \quad (74)$$

$$T : \mathcal{V}_{\text{symp}}(\Omega) \rightarrow \mathcal{L}_s(\Omega_c), \quad T(f) = \frac{f''(0)}{2} \quad (75)$$

Theorem 8.2. *The map T , given by (74), (75), establishes a projection of the classical statistical model M_{symp}^α onto the Dirac-von Neumann quantum model N_{quant} . The map (74) is one-to-one (bijection); the map (75) is only onto (surjection). The latter map is a \mathbf{R} -linear operator. Classical and quantum averages are coupled via the asymptotic equality (72).*

We remark that our projection map $T : \mathcal{V}_{\text{symp}}(\Omega) \rightarrow \mathcal{L}_s(\Omega_c)$ fulfills an important postulate for classical \rightarrow quantum correspondence which was used by J. von Neumann:

$$T\left(\sum \lambda_j f_j\right) = \sum \lambda_j T(f_j), \quad \lambda_j \in \mathbf{R}, f_j \in \mathcal{V}_{\text{symp}}(\Omega). \quad (76)$$

Here quantum observables $A_j = T(f_j)$ can be incompatible, so these operators can be noncommuting, see von Neumann [10]. This postulate was strongly criticized by J. Bell [23] and L. Ballentine [24] as nonphysical – because it is not easy to give a physical meaning to a linear combination of

incompatible observables. I agree that their arguments are not meaningless and there can be really problems with an experimental realization of the right-hand side of (76). But in a theoretical model the relation (76) might be in principle well established. Therefore, in spite the critical arguments of Bell and Ballentine, there is nothing “pathological” in this relation. We recall that our projection T is not one-to-one on the space of physical variables, but von Neumann postulated that a such correspondence should be one-to-one. Nevertheless, we have:

Corollary 8.1. *The restriction of the classical \rightarrow quantum map T onto the space of quadratic J -invariant variables $V_{\text{quad, symp}}(\Omega)$ is one-to-one map with its image $\mathcal{L}_s(\Omega_c)$.*

Remark 8.1. (Quadratic classical variables) Corollary 8.1 shows that the classical \rightarrow quantum map $T : V_{\text{quad, symp}}(\Omega) \rightarrow \mathcal{L}_s(\Omega_c)$ is nondegenerate. Each quantum observable A has uniquely defined classical preimage $f(\psi) = \frac{1}{2}(A\psi, \psi)$. In principle, we could choose the classical statistical model:

$$M_{\text{quad, symp}}^\alpha = (S_{G, \text{symp}}^\alpha(\Omega), V_{\text{quad, symp}}(\Omega)). \quad (77)$$

There is one-to-one correspondence between elements of this classical model and the Dirac-von Neumann model. Another important argument to choose this classical model is that the Schrödinger dynamics is in fact dynamics for a quadratic J -invariant Hamilton function. Nevertheless, we do not restrict our consideration to the classical model $M_{\text{quad, symp}}^\alpha$. We can speculate that linearity of the Hamilton-Schrödinger evolution is just an approximative linearity of nonlinear dynamics in Ω induced by nonquadratic Hamilton functions. But this interesting problem should be investigated in more detail.

Remark 8.2. (On the choice of a space of classical variables) We chosen the functional space $\mathcal{V}_{\text{symp}}(\Omega)$ by generalizing the class of quadratic forms $V_{\text{quad, symp}}(\Omega)$. As the main characteristic for generalization we chosen the condition of J -invariance. By Proposition 8.1 this condition implies that, for $f \in \mathcal{V}_{\text{symp}}(\Omega)$, its second derivative is a J -commuting operator. However, such a choice of the functional space of classical physical variables is not unique. There can be chosen other characteristics of quadratic forms $f \in V_{\text{quad, symp}}(\Omega)$ to obtain spaces of classical variables different from $\mathcal{V}_{\text{symp}}(\Omega)$ and, nevertheless, reproducing the class of quantum observables. The problem of an adequate choice of a space of classical variables (as well as statistical states) is not yet solved, see also section 12 for further considerations.

9 Gaussian measures inducing quantum pure states

Let $\Psi = u + iv \in \Omega_c$, so $u \in Q, v \in P$ and let $||\Psi|| = 1$. By using the conventional terminology of quantum mechanics we say that such a normalized vector of the complex Hilbert space Ψ represents a *pure quantum state*. By Born's interpretation of the wave function a pure state Ψ determines the statistical state with the density matrix:

$$D_\Psi = \Psi \otimes \Psi \quad (78)$$

This Born's interpretation of the vector Ψ – which is, on one hand, the pure state $\Psi \in \Omega_c$ and, on the other hand, the statistical state D_Ψ – was the root of appearance in QM such a notion as individual (or irreducible) randomness. Such a randomness could not be reduced to classical ensemble randomness, see von Neumann [5].

In our approach the density matrix D_Ψ has nothing to do with the individual state (classical field). The density matrix D_Ψ is the image of the classical statistical state – the J -invariant Gaussian measure ρ_Ψ on the phase space⁷ having zero mean value and the (complex) covariation operator

$$B_\Psi^c = \alpha D_\Psi \quad (79)$$

or

$$B_\Psi^c = 2\alpha \begin{pmatrix} u \otimes u + v \otimes v & v \otimes u - u \otimes v \\ u \otimes v - v \otimes u & u \otimes u + v \otimes v \end{pmatrix}.$$

In measure theory there is used the real covariation operator B . As we know (see Proposition 7.3), for a J -invariant Gaussian measure the real and complex covariation operators are coupled by the equality:

$$B_\Psi = \frac{1}{2} B_\Psi^c. \quad (80)$$

This operator has *two real eigenvectors corresponding to the same eigenvalue* $\lambda = \alpha/2$:

$$e_\Psi^{(1)} \equiv \Psi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad e_\Psi^{(2)} \equiv i\Psi = \begin{pmatrix} -v \\ u \end{pmatrix}.$$

⁷This measure is uniquely defined, see Proposition 7.3.

Thus the Gaussian measure ρ_Ψ has the support in the real plane

$$\Pi_\Psi = \{\psi = x_1 e_\Psi^{(1)} + x_2 e_\Psi^{(2)} : x_j \in \mathbf{R}\}$$

and

$$d\rho_\Psi(x_1, x_2) = \frac{1}{\pi\alpha} e^{-\frac{x_1^2 + x_2^2}{\alpha}} dx_1 dx_2.$$

We remark that for two dimensional Gaussian distributions symplectic invariance is equivalent to coincidence of eigenvalues of the covariance matrix, i.e., $(p \leftrightarrow q)$ - symmetry of Gaussian distribution: $B = \text{diag}(\alpha/2, \alpha/2)$.

Physical consequence. *There are no “pure quantum states.” States that are interpreted in the conventional quantum formalism as pure states, in fact, represent J -invariant Gaussian measures having two dimensional supports. Such states can be imagined as fluctuations of fields concentrated on two dimensional real planes of the infinite dimensional state phase-space.*

We recall that in quantum theory one distinguishes so called *pure states* and so called *mixtures*. Let us discuss this point in more detail. The set of density operators $\mathcal{D}(\Omega_c)$ is a positive cone in the space of all trace class operators: if $D_1, D_2 \in \mathcal{D}(\Omega_c)$, then $p_1 D_1 + p_2 D_2 \in \mathcal{D}(\Omega_c)$ for any $p_1, p_2 \geq 0, p_1 + p_2 = 1$. We recall, see e.g. [6], that the set of extreme points of the cone $\mathcal{D}(\Omega_c)$ coincides with the set of pure states. Thus only pure states D_ψ could not be represented in the form of a statistical mixture:

$$D = p_1 D_1 + p_2 D_2, p_j > 0, p_1 + p_2 = 1.$$

It seems that this mathematical result was one of the reasons why J. von Neumann distinguished sharply pure states and statistical mixtures and elaborated the notion of *individual randomness* - randomness associated with “pure states,” see [25]. In our approach there is no difference between “pure quantum states” and “quantum statistical mixtures” (at least from probabilistic viewpoint; geometry of distributions corresponding to “pure states” is very special; they are concentrated on two dimensional real subspaces).

Example 9.1. Let us consider the classical statistical state (Gaussian measure) $\rho \equiv \rho_\Psi$ which is projected onto the “pure quantum state” $\Psi \in \Omega, \|\Psi\| = 1$. The measure ρ is concentrated on the real plane Π_Ψ . Thus we can restrict our considerations to the phase space $\Omega = \mathbf{R} \times \mathbf{R}$ and the measure

$$d\rho(q, p) = \frac{1}{\pi\alpha} e^{-\frac{1}{\alpha}(p^2 + q^2)}.$$

Let us consider a J -invariant physical variable

$$f(q, p) = \frac{1}{2}[(p^2 + q^2) + (p^2 + q^2)^2].$$

We have

$$\langle f \rangle_\rho = \frac{1}{2} \int \int (p^2 + q^2) d\rho(q, p) + \frac{1}{2} \int \int (p^2 + q^2)^2 d\rho(q, p) = \alpha I_3 + \alpha^2 I_5,$$

where $I_n = 2 \int_0^\infty s^n e^{-s^2} ds$. Now we make the amplification of the classical variable $f_\alpha(q, p) = \frac{1}{2\alpha}[(p^2 + q^2) + (p^2 + q^2)^2]$ and obtain:

$$\langle f_\alpha \rangle_\rho = I_3 + \alpha I_5.$$

Thus approximately $\langle f_\alpha \rangle_\rho$ is equal to I_3 - the quantum average. This is the essence of quantum averaging: only the quadratic part $f_{\text{quad}}(q, p) = \frac{1}{2}(p^2 + q^2)$ of a physical variable $f(q, p)$ is taken into account; the contribution of terms of higher orders is neglected.

10 Prequantum classical statistical field theory (PCSFT)

Let $Q = P = L_2(\mathbf{R}^3)$ be the Hilbert space of real valued square integrable functions $\psi : \mathbf{R}^3 \rightarrow \mathbf{R}$ with the scalar product $(\psi_1, \psi_2) = \int_{\mathbf{R}^3} \psi_1(x) \psi_2(x) dx$. Our classical phase space $\Omega = L_2(\mathbf{R}^3) \times L_2(\mathbf{R}^3)$ consists of vector functions $\psi(x) = \begin{pmatrix} q(x) \\ p(x) \end{pmatrix}$. The symplectic operator J on this phase-space has the form:

$$q_1(x) = p(x), \quad p_1(x) = -q(x) \quad (81)$$

and the symplectic form on Ω is defined by $w(\psi_1, \psi_2) = \int_{\mathbf{R}^3} (p_2(x) q_1(x) - p_1(x) q_2(x)) dx$. The fundamental law of PCSFT is the invariance of physical variables with respect to this transformation. By introducing on Ω the canonical complex structure we obtain the $\Omega_c = L_2^{\mathbf{C}}(\mathbf{R}^3)$ - the complex Hilbert space of square integrable functions $\psi : \mathbf{R}^3 \rightarrow \mathbf{C}, \psi = q(x) + ip(x)$ with the scalar product $(\psi_1, \psi_2) = \int_{\mathbf{R}^3} \psi_1(x) \bar{\psi}_2(x) dx$. Let us consider an integral operator

$$A : \Omega \rightarrow \Omega, \quad A\psi(x) = \int_{\mathbf{R}^3} A(x, y) \psi(y) dy.$$

The kernel $A(x, y)$ of such an operator has the block structure. This operator is J -invariant iff $A_{11}(x, y) = A_{22}(x, y)$, $A_{12}(x, y) = -A_{21}(x, y)$, and it is symmetric iff $A_{11}(x, y) = A_{11}(y, x)$, $A_{12}(y, x) = A_{21}(x, y) = -A_{12}(x, y)$. The corresponding quadratic form

$$f(\psi) = \frac{1}{2} \left[\int A_{11}(x, y) \psi_1(x) \psi_2(y) dx dy + 2 \int A_{12}(x, y) \psi_2(x) \psi_1(y) \right. \\ \left. + \int A_{11}(x, y) \psi_2(x) \psi_2(y) dx dy \right]$$

Let ρ be a J -invariant measure on $\Omega = L_2(\mathbf{R}^3) \times L_2(\mathbf{R}^3)$. Its complex covariance is defined by:

$$\langle B^c \psi_1, \psi_2 \rangle = \int_{L_2(\mathbf{R}^3) \times L_2(\mathbf{R}^3)} \left(\int_{\mathbf{R}^3} \psi_1(x) \bar{\psi}(x) dx \int_{\mathbf{R}^3} \psi(x) \bar{\psi}_2(x) dx \right) d\rho(\psi).$$

Let ρ has the dispersion $\sigma^2(\rho) = \int_{L_2(\mathbf{R}^3) \times L_2(\mathbf{R}^3)} \left(\int_{\mathbf{R}^3} |\psi(x)|^2 dx \right) d\rho(\psi) = \alpha$. We find the average of the quadratic physical variable f in the state ρ :

$$\langle f \rangle_\rho = \frac{1}{2} \int_{L_2(\mathbf{R}^3) \times L_2(\mathbf{R}^3)} \left(\int A_{11}(x, y) q(x) q(y) dx dy \right. \\ \left. + 2 \int A_{12}(x, y) p(x) q(y) dx dy + \int A_{11}(x, y) p(x) p(y) dx dy \right) d\rho_B(q, p) \\ = \text{Tr} B^c A = \alpha \text{Tr} D^c \frac{f''(0)}{2},$$

where $D^c = B^c/\alpha$ is the von Neumann density operator obtained through the scaling of the covariation operator of the Gaussian measure ρ representing a classical statistical state. Since the observable is quadratic, there is the precise equality of the average of the $\frac{1}{\alpha}$ -amplification of the classical variable f and the quantum average of the self-adjoint operator $2A = f''(0)/2$.

Let us forget for a moment about mathematical difficulties and consider a singular integral operator - differential operator:

$$\mathbf{H} = -\frac{\Delta}{2m} + V(x)$$

We consider in phase-space Ω the diagonal operator $\mathbf{H}_{11} = \mathbf{H}_{22} = \mathbf{H}$, $\mathbf{H}_{12} = \mathbf{H}_{21} = 0$ or we can directly consider \mathbf{H} as acting in the complex Hilbert space $\Omega_c = L_2^C(\mathbf{R}^3)$. The corresponding classical Hamilton function is quadratic:

$$\mathcal{H}(\psi) = \frac{1}{2} \langle \mathbf{H} \psi, \psi \rangle = \frac{1}{2} \int_{\mathbf{R}^3} \left(\frac{|\nabla \psi(x)|^2}{2m} + V(x) |\psi(x)|^2 \right) dx.$$

This is the classical energy of a system with the infinite-number of degrees of freedom – the system of coupled fields $q(x)$ and $p(x)$. This system is a classical vector field; the parameter m – “mass” – is one of characteristics of this field. $\mathcal{H}(\psi)$ is an ordinary function (functional) of ψ . We can find its classical average:

$$\begin{aligned} \langle \mathcal{H} \rangle_\rho &= \frac{1}{2} \int_{L_2(\mathbf{R}^3) \times L_2(\mathbf{R}^3)} \langle \mathbf{H}\psi, \psi \rangle d\rho(\psi) \\ &= \frac{1}{2} \int_{L_2(\mathbf{R}^3) \times L_2(\mathbf{R}^3)} \left(\frac{1}{2} \int_{\mathbf{R}^3} \left(\frac{|\nabla \psi(x)|^2}{2m} + V(x)|\psi(x)|^2 \right) dx \right) d\rho_B(\psi) \\ &= \frac{1}{2} \text{Tr } B^c \left(\frac{-\Delta}{2m} + V(x) \right) = \alpha \text{Tr } D^c \mathbf{H}, \text{ where } D^c = B^c/\alpha. \end{aligned}$$

Of course, we understood that, since the operator \mathbf{H} is unbounded, the $\text{Tr } D^c \mathbf{H}$ is not well defined for an arbitrary Gaussian measure. One of possible solutions of this problem is to choose the class of Gaussian measures depending on the quantum operator. Another possibility is to follow J. von Neumann [10] and consider an approximation of \mathbf{H} by bounded operators representing *unsharp measurement* of energy.

We emphasize again that we could not guarantee that the quantum observable of energy \mathbf{H} really corresponds to a quadratic classical variable of energy $\mathcal{H}(\psi) = \langle \mathbf{H}\psi, \psi \rangle$ (in fact, to its amplification $\mathcal{H}_\alpha(\psi) = \frac{1}{2\alpha} \langle \mathbf{H}\psi, \psi \rangle$). Let us e.g. the classical energy-variable of the form:

$$\mathcal{F}(\psi) = \left[\frac{1}{2} \int_{\mathbf{R}^3} \left(\frac{|\nabla \psi(x)|^2}{2m} + V(x)|\psi(x)|^2 \right) + g \int_{\mathbf{R}^3} |\psi(x)|^4 dx \right], \quad g > 0$$

(which is J -invariant). Then it produces the same quantum average as the quadratic energy-variable:

$$\langle \mathcal{F} \rangle_\rho = \alpha \text{Tr } D^c \mathbf{H} + \alpha^2 g \int_{L_2(\mathbf{R}^3) \times L_2(\mathbf{R}^3)} \left(\int_{\mathbf{R}^3} |\psi(x)|^4 dx \right) d\rho_D(\psi)$$

Of course, the latter Hamilton function will induce nonlinear Hamiltonian dynamics in the infinite-dimensional phase-space Ω , and in principle it could be distinguished from the linear dynamics. We now consider the quadratic classical variables inducing quantum observables of the position \mathbf{x}_j and the momentum $\mathbf{p}_j (j = 1, 2, 3)$:

$$f_{\mathbf{x}_j}(\psi) = \frac{1}{2} \langle \mathbf{x}_j \psi, \psi \rangle = \frac{1}{2} \int_{\mathbf{R}^3} x_j |\psi(x)|^2 dx.$$

$$f_{\mathbf{p}_j}(\psi) = \frac{1}{2} \langle \mathbf{p}_j \psi, \psi \rangle = \frac{1}{2} \int_{\mathbf{R}^3} y_j |\tilde{\psi}(y)|^2 dy,$$

where $\tilde{\psi}(y)$ is the Fourier transform of the L_2 -function $\psi(x)$. We can also consider the quadratic classical variables inducing the angular momentum operators, e.g.,

$$f_{\mathbf{J}_z}(\psi) = \frac{1}{2} \langle \mathbf{J}_z \psi, \psi \rangle = \frac{-i}{2} \int_{\mathbf{R}^3} \left(x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial x} \right) \bar{\psi} dx dy dz$$

(but the same quantum operator also can be induced e.g. by the classical variable: $f_{\mathbf{J}_z}(\psi) = \frac{1}{2} \left(\langle \mathbf{J}_z \psi, \psi \rangle + \langle \mathbf{J}_z \psi, \psi \rangle^2 \right)$).

11 Fundamental field

In section 10 we proposed the interpretation of PCSFT by which, instead of particles, we considered corresponding fields, e.g., the electron field. Each field $\psi(x) = (q(x), p(x))$ evolves as a pair of self-inducing fields and the system of Hamiltonian equations (24) describes its motion. We consider the nonrelativistic case and scalar fields $q(x)$ and $p(x)$. In this case the Hamilton function has the form (26): $\mathcal{H}(\psi) = \int_{\mathbf{R}^6} R(x, y) \psi(x) \bar{\psi}(y) dx dy$, where

$$R(x, y) = -\frac{\nabla^2 \delta(x - y)}{2m} + \delta(x - y) V(x). \quad (82)$$

In section 10 we interpreted m as a parameter, mass, determining a scalar-complex field (or a pair of self-inducing real fields); the potential $V(x, y) = \delta(x - y) V(x)$ was considered as an external potential contributing into a self-interaction of $\psi(x)$.⁸

Such an interpretation of PCSFT was based on splitting of the kernel $R(x, y)$ into two summands, see (82), and on different interpretation of these summands. The first was considered as an internal contribution of the field and the second as an external potential.

We now propose a new interpretation. We consider also the first summand in (82) as an external potential inducing a self-interaction of the field $\psi(x)$.

⁸In fact, the component $q(x)$ self-interact with itself; the same is valid for $p(x)$; Here are no cross-interactions. This self-interaction is local, since it contains the δ -function.

Definition 11.1. *A mass interaction - field (corresponding to the mass parameter $m > 0$) is defined as*

$$R_m(x, y) = -\frac{\nabla^2 \delta(x - y)}{2m} \quad (83)$$

Fundamental Field Interpretation:

There is the unique *fundamental* field $\psi(x) = (q(x), p(x))$ which interact with various potentials.⁹ In the conventional model an interaction potential $R(x, y)$ is always of the form (26). So it contains a mass interaction field.¹⁰ Thus we propose to the following interpretation of PCSFT:

- a). There is the fundamental vector-field $\psi(x) = (p(x), q(x))$.
- b). Its internal (ontic) energy is given by:

$$\mathcal{H}(\psi) = \frac{1}{2} \|\psi\|^2 = \frac{1}{2} \int_{\mathbf{R}^3} (q^2(x) + p^2(x)) dx. \quad (84)$$

- c). There are various interaction-fields $R(x, y)$ inducing self-interactions of the fundamental field $\psi(x)$.
- d). The energy of the R -self-interacting field $\psi(x)$ is given by:

$$\mathcal{H}_R(\psi) = \frac{1}{2} \int_{\mathbf{R}^6} R(x, y) \psi(x) \bar{\psi}(y) dx dy. \quad (85)$$

For scalar fields $q(x)$ and $p(x)$ (in the nonrelativistic case) an interaction field $R(x, y)$ can always be represented in the form (82), where the first summand is referred to as the mass interaction-field.

- e). In the absence of interaction-fields the fundamental field $\psi(x)$ evolves as a system with the Hamilton function (84):

$$\dot{q} = p, \dot{p} = -q \quad (86)$$

These are oscillation of the form: $\psi(t, x) = e^{-it} \psi_0(x)$.¹¹

⁹In our mathematical model an interaction potential $R(x, y)$ can be any distribution on \mathbf{R}^6 .

¹⁰We recall that we consider nonrelativistic fields, so $m > 0$.

¹¹We remark that by the conventional interpretation of QM functions $\psi(t, x)$ for all t are just representations of the same pure state that is defined up to $\lambda = e^{i\alpha}$. But by our interpretation $\psi(t_1, x)$ and $\psi(t_2, x)$ for $t_1 \neq t_2$ are different classical fields.

e1). In the presence of an interaction-field $R(x, y)$ the fundamental field $\psi(x)$ evolves as a system with the Hamilton function (85):

$$\dot{q} = Rp, \dot{p} = -Rq, \quad (87)$$

where $R\psi(x) = \int_{R^3} R(x, y)\psi(y)dy$.

Remark 11.1. By Proposition 5.1 the quadratic form (84) of the fundamental field $\psi(x)$ is not changed in the process of the Hamiltonian evolution for any interaction-field $R(x, y)$. We call (84) *internal energy* of $\psi(x)$. In fact, we never measure the internal energy of the fundamental field $\psi(x)$. We always measure the energy of $\psi(x)$ corresponding to some interaction field $R(x, y)$.

The main difference between the fundamental ψ -field and interaction R -fields is that for the ψ -field we are not able to prepare individual states (only Gaussian distributions), but for R -fields it is possible to prepare an individual state that can be unchanged during sufficiently large interval of time. For example, we are able to prepare the Kulon potential $V(r) = \frac{e}{r}$ and not only a Gaussian ensemble of such potentials $V(r, \psi) = \frac{c(\psi)}{r}$, where ψ is a chance parameter. The same can be said about the mass field. We are able to prepare the mass potential $R_m(x, y) = \frac{\nabla^2 \delta(x-y)}{2m}$ and not only a Gaussian ensemble of such potentials $R_m(x, y) = \frac{\nabla^2 \delta(x-y)}{2m(\psi)}$ (“we are able to create a particle of the fixed maps m ”).

For the ψ -field we are not able to prepare the fixed state $\psi_0(x)$. Even when in quantum mechanics one says that “a system is in a stationary (pure) state Ψ_0 ”, in PCSFT this means just the creation of a Gaussian ensemble of Ψ -fields concentrated on the real plane $\Pi_{\Psi_0} = \{e_1 = \Psi_0, e_2 = i\Psi_0\}$.

Finally, we remark that in PCSFT there is no difference (from the physical viewpoint) between the mass potential $R_m(x, y)$ and an external potential $V(x, y)$.¹²

We finish this section with a citation from the book of Einstein and Infeld [19], p. 242-243: “ But the division into matter and field is, after the recognition of the equivalence of mass and energy, something artificial and not clearly defined. Could we not reject the concept of matter and build a pure field physics? ... There would be no place in our new physics, for both field and matter, field being the

¹²One of the purely mathematical differences is that the mass-potential $R_m(x, y) = \delta(x - y)\delta''(x)$ is more singular compared to $R_V(x, y) = \delta(x - y)V(x)$, where $V(x)$ is typically a piece wise smooth function. But, of course, there can be considered singular potentials $V(x)$, e.g., $V(x) = \delta(x)$.

only reality. This new view is suggested by the great achievements of field physics, by our success in expressing the laws of electricity, magnetism, gravitation in the form of structure laws, and finally by the equivalence of mass and energy.”

12 Dispersion preserving dynamics with non-quadratic Hamilton functions

By considering nonquadratic observables, see section 8, we come to a new interesting problem: investigation of dynamics with nonquadratic Hamilton functions. Let us consider an arbitrary Hamilton function $\mathcal{H} : \Omega \rightarrow \mathbf{R}$. The first important remark is that such a dynamics would transfer Gaussian states into Gaussian iff \mathcal{H} is quadratic.

Suppose that, for any $\psi \in \Omega$, the system of Hamiltonian equations (6) has the unique solution, $\psi(t) \equiv U_t\psi, \psi(0) = \psi$. In this case there is well defined the map (Hamiltonian flow)

$$U_t : \Omega \rightarrow \Omega. \quad (88)$$

This map induces the map U_t^* in the space of probability measures $PM(\Omega)$ on the phase-space Ω , see section 4. As was already mentioned, in the non-quadratic case the measure $U_t^*\rho, t > 0$, can be non-Gaussian even for a Gaussian measure ρ . For nonquadratic Hamilton functions we cannot restrict the classical statistical model to the model with Gaussian states. We should consider the space of statistical states consisting of all probability measures ρ on Ω that have the zero mean value and the dispersion α . Denote this class by the symbol $PM^\alpha(\Omega)$. May be we should consider the subclass $PM_{\text{symp}}^\alpha(\Omega)$ of $PM^\alpha(\Omega)$ consisting of J -invariant measures: $J^*\rho = \rho$. But at the moment we consider arbitrary measures.

We are interested in Hamiltonian dynamics U_t in the phase space Ω that induces dynamics U_t^* in $PM^\alpha(\Omega)$. Such a dynamics preserves the zero mean value and the dispersion α .

Quantum dynamics corresponding to the classical Hamiltonian dynamics with a quadratic J -invariant Hamilton functions is an example of dynamics preserving the zero mean value and the dispersion. We are interested in more general dynamics with similar features.

We find the dispersion of $U_t^*\rho$ for an arbitrary $\rho \in PM(\Omega)$ having zero

mean value:

$$\sigma^2(U_t^* \rho) = \int_{\Omega} \|U_t \psi\|^2 d\rho(\psi). \quad (89)$$

We are interested in a Hamiltonian dynamics such that dispersions of probability measures are preserved – *dispersion preserving dynamics*.

Suppose that U_t preserves the mean value of a measure. By (89) if U_t preserves the norm on the phase space Ω then U_t^* preserves the dispersion. We remark that a nonlinear norm preserving map $U : \Omega \rightarrow \Omega$ need not be one-to-one or onto. Moreover, it need not be an isometry: $\|U\psi\| = \|\psi\|$ for any $\psi \in \Omega$ does not imply that $\|U\psi_1 - U\psi_2\| = \|\psi_1 - \psi_2\|$.

It is easy to find the sufficient and necessary condition for norm-preserving dynamics induced by a Hamilton function $\mathcal{H}(\psi)$. We can write the general Hamiltonian equation (6) in the form:

$$\dot{\psi} = J\mathcal{H}'(\psi). \quad (90)$$

Theorem 12.1. *Let the flow U_t induced by a Hamilton function $\mathcal{H}(\psi)$ be a surjection, i.e., $U_t(\Omega) = \Omega$. Then it is norm preserving iff the following equality, cf. (37), section 5, holds*

$$(J\mathcal{H}'(\psi), \psi) = 0, \psi \in \Omega. \quad (91)$$

Proof. a) Let $\|U_t \psi\|^2 = \|\psi\|^2$ for any $\psi \in \Omega$. By using the representation (90) we obtain:

$$0 = \frac{d}{dt} \|U_t \psi\|^2 = 2(J\mathcal{H}'(U_t \psi), U_t \psi).$$

Thus $(J\mathcal{H}'(U_t \psi), U_t \psi) = 0, \psi \in \Omega$. Now we use the fact that $U_t(\Omega) = \Omega$ and obtain the equality (91).

b) Let the equality (91) hold for any point of $\psi \in \Omega$. Then, in particular,

$$(J\mathcal{H}'(U_t \psi), U_t \psi) = 0 \quad (92)$$

for any $\psi \in \Omega$. Thus $\frac{d}{dt} \|U_t \psi\|^2 = 0$ and hence $\|U_t \psi\| = \|\psi\|, t \geq t_0, \psi \in \Omega$.

We remark that (91) implies norm preserving even in the case when U_t is not surjection.

Denote the class of maps $f : \Omega \rightarrow \mathbf{R}$ satisfying the condition (91) by the symbol $W(\Omega)$.

Corollary 12.1. *A Hamiltonian flow is norm preserving iff the equality (92) holds.*

The equation (91) is a linear equation with respect to \mathcal{H} :

$$\left(\frac{\partial \mathcal{H}}{\partial q}, p\right) = \left(\frac{\partial \mathcal{H}}{\partial p}, q\right) \quad (93)$$

Theorem 12.2. *Let the condition $\mathcal{H} \in W(\Omega)$. Then $\mathcal{H}''(0) \in \mathcal{L}_{\text{symp},s}(\Omega)$.*

Proof. We have: $(\mathcal{H}'(\psi), J\psi) = 0$. Thus $\mathcal{H}''(\psi)J\psi + J^*\mathcal{H}'(\psi) = 0$ and, hence, $\mathcal{H}'''(\psi)J\psi + \mathcal{H}''(\psi)J + J^*\mathcal{H}''(\psi) = 0$. Therefore

$$[\mathcal{H}''(0), J] = 0. \quad (94)$$

We pay attention that in general we have:

$$[\mathcal{H}''(\psi), J] = -\mathcal{H}'''(\psi)J\psi. \quad (95)$$

We pay attention that, for any map $\mathcal{H} : \Omega \rightarrow \mathbf{R}$, we can represent

$$\mathcal{H}'' = \begin{pmatrix} \frac{\partial^2 \mathcal{H}}{\partial q^2} & \frac{\partial^2 \mathcal{H}}{\partial q \partial p} \\ \frac{\partial^2 \mathcal{H}}{\partial p \partial q} & \frac{\partial^2 \mathcal{H}}{\partial p^2} \end{pmatrix}$$

The condition $\mathcal{H}''(0, 0) \in \mathcal{L}_{\text{symp},s}(\Omega)$ implies that

$$\frac{\partial^2 \mathcal{H}}{\partial q^2}(0, 0) = \frac{\partial^2 \mathcal{H}}{\partial p^2}(0, 0), \quad \frac{\partial^2 \mathcal{H}}{\partial q \partial p} = -\frac{\partial^2 \mathcal{H}}{\partial p \partial q}. \quad (96)$$

The latter equality should not be surprising even in the light of the well known equality of mixed partial derivatives for any two times continuously differentiable map. Of course, we always have:

$$\frac{\partial^2 \mathcal{H}}{\partial p_i \partial q_j} = \frac{\partial^2 \mathcal{H}}{\partial q_j \partial p_i}$$

for any i, j . Let us consider an illustrative example. Let us consider the quadratic Hamilton function: $\mathcal{H}(q_1, q_2, p_1, p_2) = p_1 q_2 - q_1 p_2$. Here we have:

$$\frac{\partial^2 \mathcal{H}}{\partial q \partial p} = \begin{pmatrix} \frac{\partial^2 \mathcal{H}}{\partial q_1 \partial p_1} & \frac{\partial^2 \mathcal{H}}{\partial q_1 \partial p_2} \\ \frac{\partial^2 \mathcal{H}}{\partial q_2 \partial p_1} & \frac{\partial^2 \mathcal{H}}{\partial q_2 \partial p_2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

and

$$\frac{\partial^2 \mathcal{H}}{\partial p \partial q} = \begin{pmatrix} \frac{\partial^2 \mathcal{H}}{\partial p_1 \partial q_1} & \frac{\partial^2 \mathcal{H}}{\partial p_1 \partial q_2} \\ \frac{\partial^2 \mathcal{H}}{\partial p_2 \partial q_1} & \frac{\partial^2 \mathcal{H}}{\partial p_2 \partial q_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We remark that any polynomial of type considered in Example 8.1 satisfies the condition (91). Therefore each Hamilton function of such a type induces the flow $U_t(\psi)$ that preserves the norm, e.g., $\mathcal{H}(\psi) = a_1(\mathbf{H}\psi, \psi) + a_2(\mathbf{H}\psi, \psi)^2$, where $[\mathbf{H}, J] = 0$. But we do not know general relation between the functional classes of J -invariant functions and functions satisfying (91).

On the other hand, by using the condition (91) we can easily find Hamilton functions that induce flows which do not preserve the norm. Let us consider (in the two dimensional case) the map $\mathcal{H}(q, p) = q^2 p$. For this map the condition (91) does not hold true. Therefore the Hamiltonian flow corresponding to this map does not preserve the norm.

We now investigate conditions for preserving of the average. There is given a measure ρ with zero mean value ("fluctuation of vacuum"): $m_\rho = 0$. We would like to find a sufficient condition for preserving of this value: $m_{\rho_t} = 0$ for $t \geq 0$. Let us consider the class of symmetric measures: such ρ that $g_{-1}^* \rho = \rho$, where $g_{-1}\psi = -\psi$. We remark that any even measure has the zero mean value.

Proposition 12.1. *Let ρ be a symmetric measure and let $U_t(\psi)$ be an odd Hamiltonian flow:*

$$U_t(-\psi) = -U_t(\psi). \quad (97)$$

Then the average $m_{\rho_t} = 0$ for $t \geq 0$.

We even can prove that:

Proposition 12.2. *An odd Hamiltonian flow preserves the class of symmetric measures.*

Proof. We should get $g_{-1}^* U_t^* \rho = U_t^* \rho$. We have:

$$\begin{aligned} \int f(\psi) dg_{-1}^* U_t^* \rho(\psi) &= \int f(U_t(-\psi)) d\rho(\psi) = \int f(-U_t(\psi)) d\rho(\psi) \\ &= \int f(U_t(\psi)) d\rho(\psi) = \int f(\psi) dU_t^* \rho(\psi). \end{aligned}$$

Proposition 12.3. *Let the Cauchy problem for a Hamiltonian equations be well posed. Then the Hamiltonian flow is odd if \mathcal{H}' is odd.*

Proof. a). Let (97) hold. Then $\frac{dU_t}{dt}(\psi) = -\frac{dU_t}{dt}(-\psi)$. Thus $\mathcal{H}'(U_t(\psi)) = -\mathcal{H}'(U_t(-\psi))$. Hence

$$\mathcal{H}'(\phi) = -\mathcal{H}'(-\phi) \quad (98)$$

for any $\phi = U_t\psi$. Since the problem is well posed, any $\phi \in \Omega$ can be represented in this form.

b). Let now (98) hold. We have: $-\frac{dU_t}{dt}(-\psi) = -J\mathcal{H}'(U_t(-\psi)) = J\mathcal{H}'(-U_t(-\psi))$. But the problem is well posed, so the solution is unique. Thus (97) holds.

Corollary 12.2. *Let the Hamilton function $\mathcal{H}(\psi)$ be J -invariant. Then its flow preserves the averages of symmetric measures.*

Finally, we pay attention that any J -invariant measure is symmetric (and in particular its average is zero).

Corollary 12.3. *Let the Hamilton function $\mathcal{H}(\psi)$ and the measure ρ be J -invariant. Then the Hamiltonian flow preserves the (zero) mean value of ρ .*

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